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REYNOLDS AVERAGED NAVIER-STOKES EQUATIONS (RANS) AND TURBULENCE MODELS

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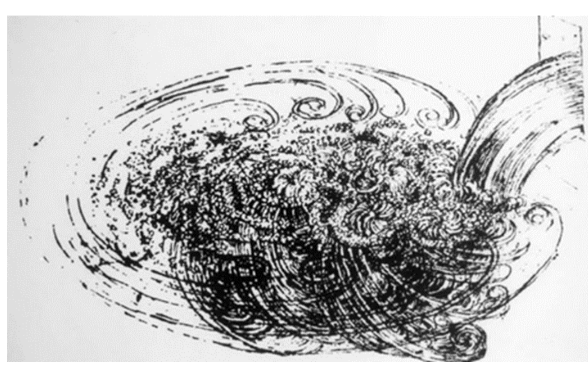
INTRODUCTION

Leonardo da Vinci (1452-1519) was the first one to use the word turbulence (*turbolenza*) to describe the vortical irregular motion of a fluid flow



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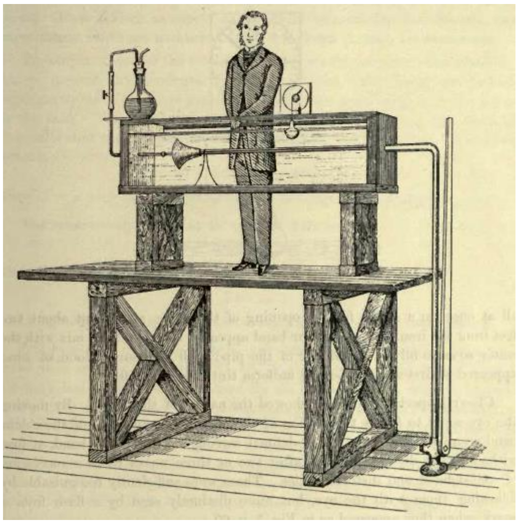
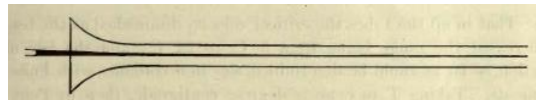
SOME PRELIMINARIES

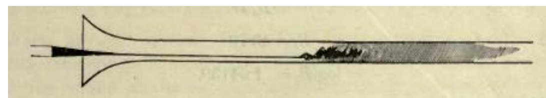
REYNOLD'S EXPERIMENT (1883)



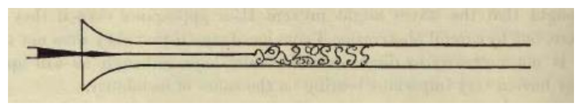
Experimental set-up of Reynold´s experiment, taken from his paper from 1883.



Flow pattern of the dye at low velocities. "Direct" motion or laminar regime.



Flow pattern of the dye at high velocities. "Sinuous" motion or turbulent regime.



Flow pattern of the dye in the "sinuous" motion or turbulent regime when illuminated with a sparkling light. Reynolds observed the eddy or curly motion of the fluid.

FIGURES FROM REYNOLDS' PAPER

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SOME PRELIMINARIES

Newton's second law for an incompressible Newtonian fluid in a gravitational field, is written as:

Component x:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

Component y:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

Component z:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

Or, in vectorial form:

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=-\nabla p+\mu\nabla^2\vec{v}+\rho\vec{g}$$

NAVIER-STOKES EQUATIONS

SOME PRELIMINARIES

Formally, the problem is solved: We have 4 differential equations (with its initial and boundary conditions) and 4 unknowns:

Continuity:

$$\nabla \cdot \vec{v} = 0$$

Navier-Stokes equations (3):

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

Unknowns: p, u, v, w

This set of equations do not have restrictions regarding the flow regime. Formally, they can be applied any flow regime: laminar, turbulent or transitional laminar-turbulent.

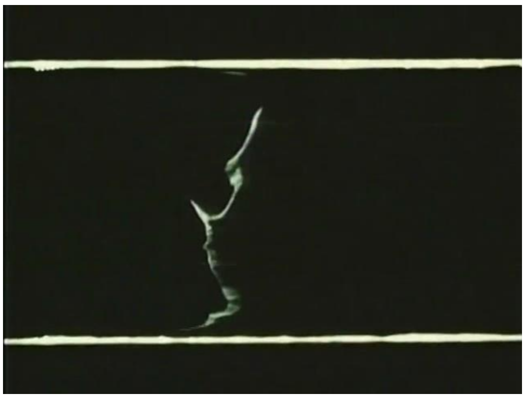
SOME PRELIMINARIES

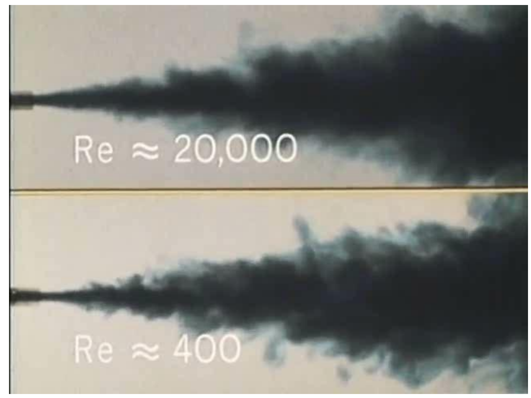
The equations of Navier-Stokes are extremely difficult to solve, except for some simple particular cases. Although they are valid for any flow regime, they are only applied to laminar flows to get analytical solutions.

This limitation was not known when Navier (1822) and Stokes (1845) published their equations (at that time, laminar and turbulent regimes where not identified yet).

SOME PRELIMINARIES
PRESENCE OF EDDIES IN TURBULENT FLOWS

The existence of eddies in turbulent flows was recognized in the XIX century.





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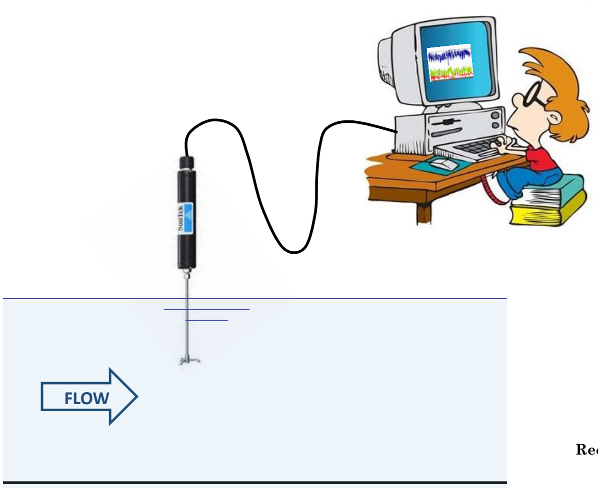
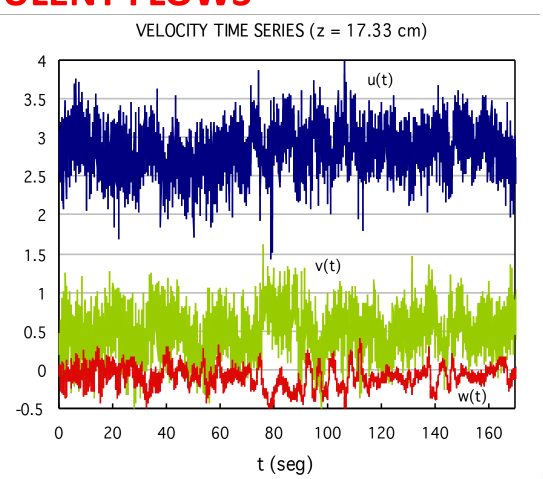
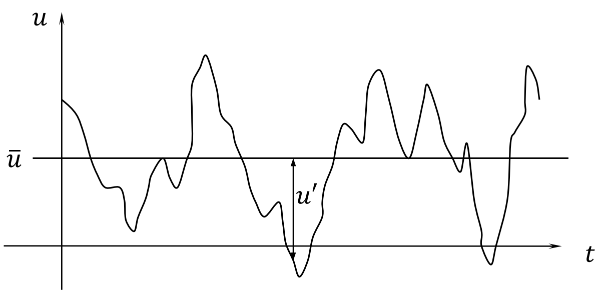
SOME PRELIMINARIES
VELOCITY MEASUREMENTS IN TURBULENT FLOWS





Record of the three components of the velocity measured at one location in a turbulent flow in an open channel

SOME PRELIMINARIES
REYNOLDS' DECOMPOSITION



$$\vec{V}(\vec{x},t) = \overline{\vec{V}}(\vec{x}) + \vec{V}'(\vec{x},t)$$

INSTANSTANEOUS VELOCITY

AVERAGE COMPONENT

FLUCTUATING COMPONENT

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$
$$p = \bar{p} + p'$$

SOME PRELIMINARIES
REYNOLDS' EQUATIONS

Replacing the decomposed velocities in the continuity equation:

$$\frac{\partial(\bar{u}+u')}{\partial x} + \frac{\partial(\bar{v}+v')}{\partial y} + \frac{\partial(\bar{w}+w')}{\partial z} = 0$$

Averaging over the turbulence, we get:

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z} = 0$$

CONTINUITY EQUATION IS SATISFIED FOR THE AVERAGE COMPONENTS OF THE VELOCITIES

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

CONTINUITY EQUATION IS SATISFIED FOR THE FLUCTUATING COMPONENTS OF THE VELOCITIES

SOME PRELIMINARIES

REYNOLDS' EQUATIONS

Replacing the decomposed velocities in the x component of the Navier-Stokes equations and after some algebra which is detailed in the class notes the following equations is obtained:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}+\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\rho g_x$$

After obtaining this equation, Reynolds had a bright idea that allowed him to interpret the terms arising from the velocity fluctuations: $\overline{u'^2}, \overline{u'v'}, \overline{u'w'}$.

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SOME PRELIMINARIES

REYNOLDS' EQUATIONS

Thus the x component of the averaged Navier-Stokes equation becomes:

$$\rho\left(\frac{d\bar{u}}{dt}+\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\rho g_x$$

To give the form of:

$$\rho\frac{d\bar{u}}{dt}=\bar{F}_{Vx}$$

$$\rho\left(\frac{d\bar{u}}{dt}+\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\rho g_x$$

$$\rho\frac{d\bar{u}}{dt}=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}+\rho g_x-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

BRIGHT IDEA!!!

SOME PRELIMINARIES
REYNOLDS' EQUATIONS

$$\Sigma\overline{F}_{Vx} = -\frac{\partial \overline{p}}{\partial x} + \mu\nabla^2\overline{u} + \rho g_x - \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

FORCES ACTING IN THE VOLUME OF FLUID = **FORCES DUE TO PRESSURE** + **FORCES DUE TO VIASCOUS STRESSES** + **FORCE DUE TO GRAVITY** + **APARENT FORCES ORIGINATED BY THE TURBULENCE**

APARENT FORCES :
THEY ARE NOT REALLY FORCES.
THEY ARE FLUXES OF MOMENTUM DUE TO THE TURBULENT FLUCTUATIONS WHICH ARE INTERPRETED AS FORCES (REYNOLD'S GREAT IDEA)

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SOME PRELIMINARIES
REYNOLDS' EQUATIONS

Defining the turbulent or Reynolds stresses as:

$$\tau_{Txx} = -\rho\overline{u'^2}\,, \quad \tau_{Tyx} = -\rho\overline{u'v'}\,, \quad \tau_{Tzx} = -\rho\overline{u'w'}$$

The viscous stresses are:

$$\tau_{Vxx} = 2\mu\frac{\partial \bar{u}}{\partial x}\,, \quad \tau_{Vyx} = \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right), \quad \tau_{Vzx} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)$$

The total stresses are:

$$T_{xx} = \tau_{Vxx} + \tau_{Txx}\,, \quad T_{yx} = \tau_{Vyx} + \tau_{Tyx}\,, \quad T_{zx} = \tau_{Vzx} + \tau_{Tzx}$$

SOME PRELIMINARIES
REYNOLDS' EQUATIONS

x component of the momentum equation:

$$\rho\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)=-\frac{\partial\bar{p}}{\partial x}+\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{yx}}{\partial y}+\frac{\partial T_{zx}}{\partial z}+\rho g_x$$

In the same way:

y component of the momentum equation:

$$\rho\left(\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{w}\frac{\partial\bar{v}}{\partial z}\right)=-\frac{\partial\bar{p}}{\partial y}+\frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{zy}}{\partial z}+\rho g_y$$

z component of the momentum equation:

$$\rho\left(\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right)=-\frac{\partial\bar{p}}{\partial z}+\frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{yz}}{\partial y}+\frac{\partial T_{zz}}{\partial z}+\rho g_z$$

THESE ARE THE REYNOLDS EQUATIONS FOR THE TURBULENT FLOW PUBLISHED IN 1895

In general:

$$T_{ij}=\tau_{Vij}+\tau_{Tij}$$

$$\tau_{Vij}=\mu\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)$$

$$\tau_{Tij}=-\rho\overline{u_i'u_j'}$$

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SOME PRELIMINARIES
THE PROBLEM OF TURBULENCE CLOSURE

Although the Reynolds equations are an important step in the study of the turbulence, they do not solve the problem.

We have 4 differential equations: Continuity (1) and Reynolds' equations (3)

We have 10 unknowns: $\bar{u},\bar{v},\bar{w},\bar{p},\overline{u'^2},\ \overline{v'^2},\overline{w'^2},\overline{u'v'}\ ,\overline{u'w'}\ ,\overline{v'w'}$

We need relationships for the Reynolds stresses: $-\rho\overline{u_i'u_j'}$

As there is not a theory based only on the first principles of the physics, all the available models necessarily require some experimental data.

SOME PRELIMINARIES

BOUSSINESQ'S CLOSURE OF THE TURBULENCE: EDDY VISCOSITY

Boussinesq proposed his model in 1877 (almost 20 years before that the Reynolds' equations were published).

The viscous stresses are given by:

$$\tau_{Vij} = \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

In analogy to the viscous stresses, Boussinesq proposed an eddy viscosity coefficient ε such that

$$\tau_{Tij} = \varepsilon\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

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SOME PRELIMINARIES

BOUSSINESQ'S CLOSURE OF THE TURBULENCE: EDDY VISCOSITY

$$\tau_{Vij} = \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \quad , \quad \tau_{Tij} = \varepsilon\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

There is a strong difference between μ and ε:

- *The dynamic viscosity μ is a property of the fluid*
- *The eddy viscosity ε is a property of the flow*
- In the turbulent region of the flow $\mu \ll \varepsilon$

For open channel flows, Boussinesq proposed:

$$\varepsilon = \rho g A h u_0$$

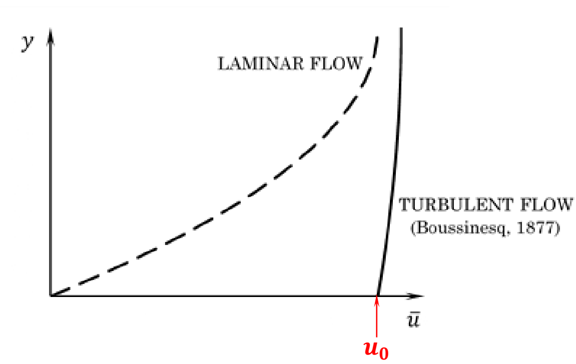
- A: COEFFICIENT THAT DEPENDS ON THE WALL ROUGHNESS
- h: FLOW DEPTH
- u_0: VELOCITY AT THE WALL

SOME PRELIMINARIES

BOUSSINESQ'S CLOSURE OF THE TURBULENCE: EDDY VISCOSITY

According to Boussinesq, the velocity distribution in a 2-D uniform, permanent open-channel flow is:

$$\bar{u} = u_0 + \frac{\sin\theta}{Ahu_0}\left(Hy - \frac{1}{2}y^2\right)$$



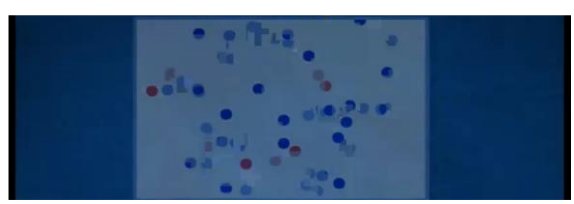
Boussinesq's result does not satisfy the non-slip condition!!

SOME PRELIMINARIES

PRANDTL'S CLOSURE OF THE TURBULENCE: MIXING LENGTH

Ludwig Prandtl (1875-1953) is among the greatest researchers in fluid mechanics of the XX century. His most important contribution is his boundary layer theory (1904), by which he was nominated to the Nobel prize in 1928 (he did not get it).

In his closure of the turbulence problem, Prandtl made an analogy with the kinetic theory of gases, according to which the gas molecules can travel, preserving its momentum, until they collides with other.

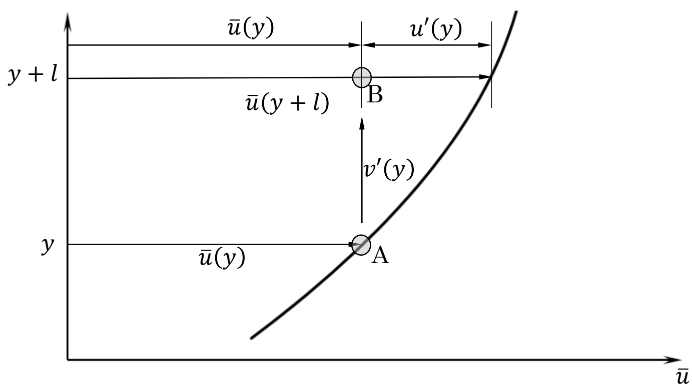


The average length that the molecules travel before colliding is named "free mean path"

SOME PRELIMINARIES

PRANDTL'S CLOSURE OF THE TURBULENCE: MIXING LENGTH

The idea behind the mixing length:

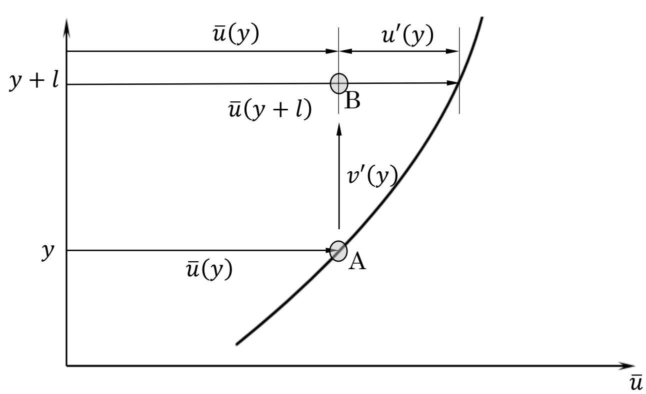


In analogy to the kinetic theory of gases, according to Prandtl, parcels of fluid are displaced *without loosing their identity* (i.e., they preserve their momentum) due to the turbulent fluctuations.

The average distance travelled by the parcels of fluids is denominated *mixing length* (l)

SOME PRELIMINARIES

PRANDTL'S CLOSURE OF THE TURBULENCE: MIXING LENGTH



A parcel of fluid, initially in A (located at y), due to the fluctuation of the vertical velocity v', moves to B (located at $y+l$), preserving its initial momentum $\rho\bar{u}(y)$, which is imposed at the new location. Thus, the new velocity at $y+l$ is $\bar{u}(y)$.

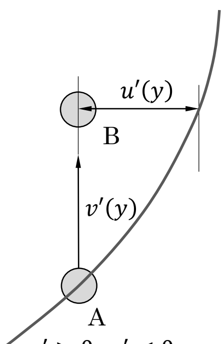
At $y+l$, the instantaneous change from $\bar{u}(y+l)$ to $\bar{u}(y)$ corresponds to the velocity fluctuation $u'(y)$.

$$u' = \bar{u}(y) - \bar{u}(y+l)$$

SOME PRELIMINARIES

PRANDTL'S CLOSURE OF THE TURBULENCE: MIXING LENGTH

$$u' = \bar{u}(y) - \bar{u}(y + l)$$



$v' > 0\,, u' < 0 \;\rightarrow u'v' < 0$

$v' < 0\,, u' > 0 \;\rightarrow u'v' < 0$

Expanding in Taylor's series:

$$u' = \bar{u}(y) - \left(\bar{u}(y) + \frac{\partial \bar{u}}{\partial y} l + \cdots\right) \rightarrow \; u' = -l \frac{\partial \bar{u}}{\partial y}$$

Experimental evidence: $|u'| \sim |v'|$

Thus :

$$v' \sim l \frac{\partial \bar{u}}{\partial y}$$

$$\tau_{Txy} = -\rho \overline{u'v'} \rightarrow \boldsymbol{\tau_{Txy} = \rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2}$$

We cannot go further in the analysis. We need an expression for the mixing length l

BOUSSINESQ AND PRANDTL MODELS

The models before presented have serious limitations in their application and most of the time they cannot applied to conditions different to those that they were developed.

However, they are the base of more general models.

SCALES OF TURBULENCE

Before presenting closure models, it seems natural to ask: Why do not solve numerically the continuity and Navier-Stokes equations?

$$\nabla \cdot \vec{v} = 0$$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

Numerical solution of the above equations for turbulent flows is a valid strategy, but *it requires a discretization smaller than the smallest eddy* of the turbulent flow.

What is the smallest scale associated to the turbulence?

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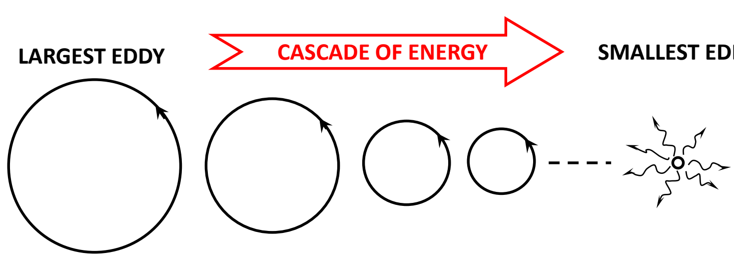
KOLMOGOROV'S SCALES

Turbulent flows convey eddies of many sizes. The largest, obviously, scales with the dimension of the flow domain. There is a continuum of eddy sizes from the largest to the smallest, which dissipates the energy due to the fluid viscosity.



KOLMOGOROV'S SCALES

Kolmogorov (1941) proposed that the scales of the smallest eddy depends on the kinematic viscosity (ν) and the energy dissipation rate per unit mass (ε).

Dimensions: $[\nu] = \mathrm{L}^2\mathrm{T}^{-1}$, $\varepsilon = \mathrm{L}^2\mathrm{T}^{-3}$. Using dimensional analysis is easy to get:

LENGTH SCALE: $\eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ ***EDDY SIZE***

TIME SCALE: $\tau_K = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$ ***TURNOVER TIME***

VELOCITY SCALE: $v_K = (\nu\varepsilon)^{1/4}$ ***EDDY VELOCITY***

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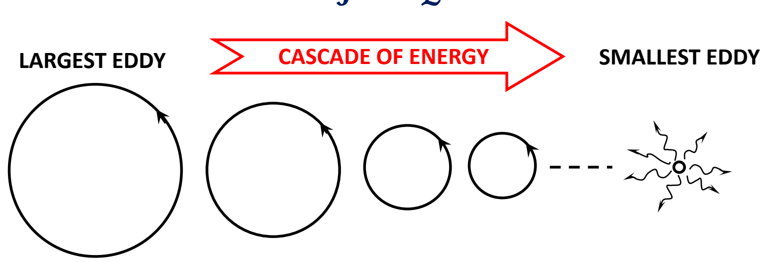
KOLMOGOROV'S SCALES

$$\mathbf{PRODUCTION}\ (P) = \mathbf{DISIPATION}\ (\varepsilon)$$

$$P \sim \frac{\mathcal{U}^2}{\mathcal{T}} = \frac{\mathcal{U}^3}{\mathcal{L}}$$



$$P \qquad = \qquad \varepsilon$$

$$\varepsilon \sim \frac{\mathcal{U}^3}{\mathcal{L}}$$

VISCOUS DISSIPATION DEPENDS ON THE DYNAMICS OF THE LARGE SCALES (DOES NOT INVOLVES VISCOSITY)

KOLMOGOROV'S SCALES

$$\varepsilon \sim \frac{\mathcal{U}^3}{\mathcal{L}}$$

LENGTH SCALE: $\eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim \left(\frac{\nu^3}{\mathcal{U}^3\mathcal{L}^3}\right)^{1/4} \mathcal{L} \quad \rightarrow \quad \eta_K = Re^{-3/4}\mathcal{L}$

TIME SCALE: $\tau_K = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \quad \rightarrow \quad \tau_K = Re^{-1/2}\mathcal{T}$

VELOCITY SCALE: $v_K = (\nu\varepsilon)^{1/4} \quad \rightarrow \quad v_K = Re^{-1/4}\mathcal{U}$

KOLMOGOROV'S SCALES

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To have an idea of the magnitude of the scales, let's consider the flow in an open channel.
Water depth: 0.3 m, channel width: 0.5 m, slope: 0.0001, Manning's *n*: 0.014 → V ~ 0.2 m/s

Estimating $\mathcal{L} \sim 0.3$ m, $\mathcal{U} \sim V \sim 0.2$ m/s, $\nu \sim 10^{-6}$ m²/s → $Re \sim 6 \times 10^4$

$$\eta_K = Re^{-3/4}\mathcal{L} \sim 0.08 \text{ mm} \qquad \tau_K = Re^{-1/2}\mathcal{T} \sim 0.006 \text{ s}$$

A numerical solution of the Navier-Stokes equations (Direct Numerical Simulation) requires a discretization such that:

$$\Delta x \sim \Delta y \sim \Delta y < \frac{\eta_K}{2} = 0.04 \text{ mm} \quad , \quad \Delta t < \frac{\tau_K}{2} = 0.003 \text{ s}$$

DNS IS NOT PRACTICAL FOR ENGINEERING APPLICATIONS YET

REYNOLDS' EQUATION IN INDEX NOTATION (EINSTEIN CONVENTION)

Continuity and Reynolds equations are written in a compact form using a index notation:

CONTINUITY $$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

MOMENTUM (REYNOLDS) EQUATIONS $$\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{T_{ij}}{\rho}\right)$$

$$T_{ij} = -\bar{p}\delta_{ij} + 2\mu S_{ij} - \rho\overline{u_i' u_j'} \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

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KINETIC ENERGY EQUATION

The equation for the instantaneous kinetic energy is obtained making: $(N - S\ eq.) \cdot \vec{V}$

Making the Reynolds' decomposition of the kinetic energy equation and taking the average, the equation for the mean kinetic energy and turbulent kinetic energy are obtained:

MEAN KINETIC ENERGY $$\rho\bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) = \frac{\partial}{\partial x_j}\left(T_{ij}\bar{u}_i\right) - T_{ij}S_{ij}$$

$$\bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) = \frac{\partial}{\partial x_j}\left(-\frac{p}{\rho}\bar{u}_j + 2\nu\bar{u}_i S_{ij} - \overline{u_i' u_j'}\bar{u}_i\right) - 2\nu S_{ij}S_{ij} + \overline{u_i' u_j'} S_{ij}$$

- $\bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right)$: RATE OF CHANGE OF THE KINETIC ENERGY OF THE MEAN FLOW
- $-\frac{p}{\rho}\bar{u}_j$: PRESSURE WORK
- $2\nu\bar{u}_i S_{ij}$: TRANSPORT OF THE KINETIC ENERGY OF THE MEAN FLOW BY VISCOUS STRESSES
- $-\overline{u_i' u_j'}\bar{u}_i$: TRANSPORT OF THE KINETIC ENERGY OF THE MEAN FLOW BY REYNOLDS STRESSES
- $-2\nu S_{ij}S_{ij}$: VISCOUS DISSIPATION
- $\overline{u_i' u_j'} S_{ij}$: PRODUCTION OF TURBULENT KINETIC ENERGY

IN MOST FLOWS THESE VISCOUS TERMS ARE NEGLEGIBLE

TURBULENT KINETIC ENERGY

TURBULENT KINETIC ENERGY

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

$$\bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i' u_i'}\right) = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j' p'} + \frac{1}{2}\overline{u_i' u_i' u_j'} - 2\nu\overline{u_i' s_{ij}}\right) - \overline{u_i' u_j'} S_{ij} - 2\nu\overline{s_{ij} s_{ij}}$$

RATE OF CHANGE OF THE TURBULENT KINETIC ENERGY

PRESSURE WORK

TRANSPORT OF TURBULENT KINETIC ENERGY BY VELOCITY FLUCTUATIONS

TRANSPORT OF TURBULENT KINETIC ENERGY BY VISCOUS EFFECTS

PRODUCTION OF TURBULENT KINETIC ENERGY

ε
VISCOUS DISSIPATION

DEFORMATION WORK (MOST IMPORTANT TERMS)

$$k \equiv \frac{1}{2}\overline{u_i' u_i'} = \frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2}$$

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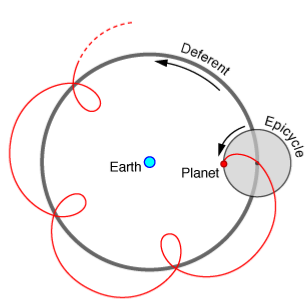
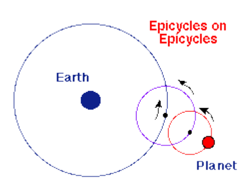
TURBULENCE MODELS

Turbulence modelling has been compared with the model of epicycles and deferents used in astronomy from ancient Greeks until the middle ages to explain the retrograde motion of the planets. Astronomers were constantly adding epicycles over epicycles to adjust the motions to new data.





TURBULENCE MODELLING DOES NOT CONTRIBUTE WITH NEW KNOWLEDGEMENT OF THE PHYSICS OF TURBULENCE

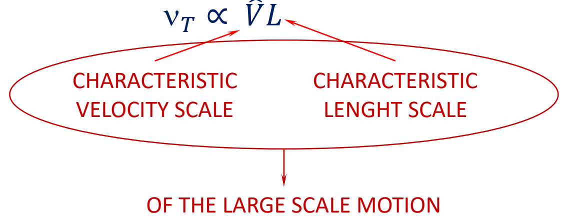
TURBULENCE MODELS

In general, the goal of all the models is to find the eddy viscosity that will permit to compute the Reynolds stresses $-\rho\overline{u_i' u_j'}$.

The concept of turbulent viscosity (or eddy viscosity) is the basis of all the models.

ν_T: (kinematic) eddy viscosity [L^2T^{-1}]

$$\nu_T \propto \hat{V} L$$



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TURBULENCE MODELS

THE CONCEPT OF TURBULENT VISCOSITY

REYNOLDS EQUATIONS:

$$\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{-\bar{p}}{\rho}\delta_{ij} + \nu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \overline{u_i' u_j'}\right)$$

$$-\overline{u_i' u_j'} = \nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij}$$

TURBULENCE MODELS

CLASIFICATION

MODEL	CHARACTERISTICS
ZERO –EQUATION MODELS	Algebraic relationships model the eddy viscosity
"HALF-EQUATION" MODELS	An ordinary differential equation is required to be solved
ONE EQUATION MODELS	One partial differential equation is used in the model
TWO EQUATION MODELS	Two partial differential equations are used in the model

TURBULENCE MODELS

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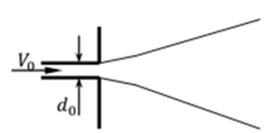
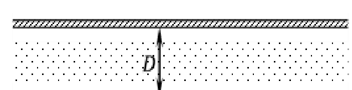
ZERO EQUATION MODELS

- **CONSTANT TURBULENT VISCOSITY.** The original Boussinesq model falls in this category: $\nu_T = gAhu_0$
  - Axisymmetric jet $\nu_T = 0.013 V_0 d_0$



  - Central region of the flow in a pipe

$$\frac{\nu_T}{\nu} = \frac{C}{2} Re \sqrt{\frac{f}{8}}$$



$$C \approx 0.07 \quad ; \quad Re = \frac{UD}{\nu}$$

TURBULENCE MODELS

ZERO EQUATION MODELS

- **TURBULENT VISCOSITY AVERAGED ALONG THE VERTICAL.**
  - For example, for a slender open channel flow:

$$\nu_T = \frac{1}{6}\kappa u_* h$$

- **PRANDTL'S MODEL FOR FREE-SHEAR LAYERS**

$$\nu_T = C\delta |U_{max} - U_{MIN}|$$

TYPE OF FLOW	PLANE MIXING LAYER	AMBIENT FLUID AT REST			PLANE WAKE
		PLANE JET	AXISYMMETRIC JET	RADIAL JET (FAN)	
C	0.01	0.014	0.011	0.019	0.026

TURBULENCE MODELS

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ZERO EQUATION MODELS

- **MIXING LENGTH MODELS**
  - The turbulent viscosity is computed from:

$$\nu_T = l^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

MIXING LENGTH IN FREE SHEAR FLOWS:

TYPE OF FLOW	PLANE MIXING LAYER	AMBIENT FLUID AT REST			PLANE WAKE
		PLANE JET	AXISYMMETRIC JET	RADIAL JET (FAN)	
$\frac{l}{\delta}$	0.7	0.9	0.75	0.125	0.16

δ: boundary layer thickness

TURBULENCE MODELS

ZERO EQUATION MODELS

- MIXING LENGTH MODELS

BOUNDARY LAYER WITH SOLID FRONTIERS:

PIPES (Nikuradse):

$$\frac{l}{R} = 0.14 - 0.08\left(1-\frac{y}{R}\right)^2 - 0.06\left(1-\frac{y}{R}\right)^4$$

NEAR THE WALL (van Driest):

$$l = y\left[1-\exp\left(-\frac{1}{A}\frac{yu_*}{\nu}\right)\right] \quad , \quad A = 26$$

von KÁRMÁN SIMILARITY RULE:

$$l = \frac{\dfrac{d\bar{u}}{dy}}{\dfrac{d^2\bar{u}}{dy^2}}$$

There is a similarity of the turbulence fluctuations in all the flow domain . It works well near the wall. Fails in jets and wakes (In inflexion points $l \to \infty$)

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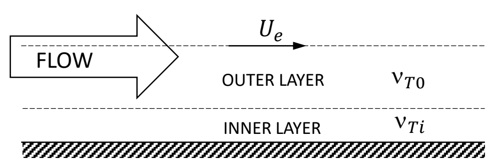
TURBULENCE MODELS

ZERO EQUATION MODELS

- MIXING LENGTH MODELS

TWO LAYER FLOWS:



CEBECI and SMITH (1974)

$$\nu_{Ti} = l^2\sqrt{\frac{\partial \bar{u}_i}{\partial x_j}\frac{\bar{u}_i}{\partial x_j}} \qquad \nu_{T0} = 0.0168 U_e \delta^* F_K$$

INTERMITENCY FUNCTION

BALDWIN and LOMAX (1978)

WAKE FUNCTION

$$\nu_{Ti} = l^2\sqrt{2W_{ij}W_{ij}} \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right) \qquad \nu_{T0} = 0.0269 F_{wk} F_K$$

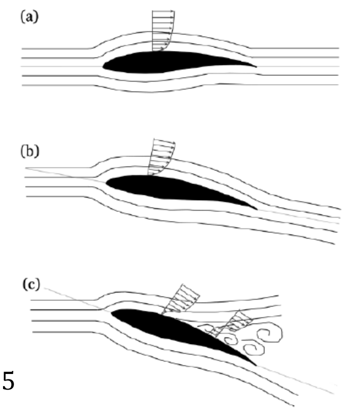
$$l = y\left[1-\exp\left(-\frac{1}{A}\frac{yu_*}{\nu}\right)\right] \quad , \quad A = 26$$

TURBULENCE MODELS

"HALF-EQUATION" MODELS

The model receives its name because it does not involve partial differential equations. It includes one ordinary differential equation.

Johnson and King (1985) developed the model to solve boundary layer flows in strong adverse pressure gradients. In this model, the advection effects are important, whereas turbulent transport and diffusion effects are much less important.



$$\nu_T = \nu_{T0}[1 - \exp(\nu_{Ti}/\nu_{T0})]$$

$$\nu_{Ti} = l^2 \frac{\sqrt{\tau_{xym}}}{\kappa y} \qquad l = y\left[1 - \exp\left(-\frac{1}{A}\frac{yu_*}{\nu}\right)\right] \quad , \quad A = 15$$

$$\nu_{T0} = 0.0269 F_{wk} F_K \sigma(x)$$

m: maximum value of τ_{xy}

FUNCTION THAT IS ADJUSTED AT EACH LOCATION x SUCH THAT:

$$\nu_{Tm} = -\frac{\tau_{xym}}{\partial \bar{u}/\partial y_m}$$

$$\bar{u}_m \frac{d\tau_m}{dx} = b_{12}\left(\sqrt{\tau_{meq}} - \sqrt{\tau_m}\right)\frac{\tau_m}{L_m} - C_{dif}\frac{\tau_m^{3/2}}{0.7\delta - y_m}\left[1 - \sqrt{\sigma(x)}\right]$$

τ_m for $\sigma(x) = 1$

DISSIPATION LENGTH SCALE $L_m = \kappa y$; $L_m = 0.09\delta$

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TURBULENCE MODELS

TRANSPORT EQUATION OF A PROPERTY

Before presenting the one- and two-equation models, it is worth to remember the form of the equations that describe the transport of a property by the flow. Let's call Γ the transported property (it can be scalar or vector). The variation of Γ is given by:

$$\frac{d\Gamma}{dt} = \frac{\partial}{\partial x_i}\left(D\frac{\partial \Gamma}{\partial x_i}\right) + S$$

In vectorial notation:

$$\frac{\partial \Gamma}{\partial t} + \vec{V} \cdot \nabla\Gamma = \nabla \cdot (D\nabla\Gamma) + S$$

where D is a diffusion coefficient and S is a source or sink term.

TURBULENCE MODELS

TRANSPORT EQUATION OF A PROPERTY

$$\frac{\partial \Gamma}{\partial t} + \vec{V} \cdot \nabla \Gamma = \nabla \cdot (D \nabla \Gamma) + S$$

Examples:

If Γ is the mass of a conservative ($S = 0$) substance dissolved in the fluid, we have the mass diffusion equation (D constant):

$$\frac{\partial c}{\partial t} + \vec{V} \cdot \nabla c = D \nabla^2 c$$

If Γ is the momentum ($\Gamma = \rho \vec{V}$)

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V} + \vec{g}$$

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = \nu \nabla^2 \vec{V} - \frac{1}{\rho} \nabla p + \vec{g}$$

$D = \nu$ coefficient of momentum diffusion

$S = -\frac{1}{\rho} \nabla p + \vec{g}$: source of momentum

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TURBULENCE MODELS

ONE EQUATION MODELS

In the one equation models a transport equation is introduced, with an algebraic relation for the turbulent length scale L.

TRANSPORT OF THE TURBULENT KINETIC ENERGY k:

$$\bar{u}_j \frac{\partial k}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j' p'} + \frac{1}{2}\overline{u_i' u_i' u_j'} - 2\nu \overline{u_i' s_{ij}}\right) - \overline{u_i' u_j'} S_{ij} - 2\nu \overline{s_{ij} s_{ij}}$$

We want to give it the form $\frac{d\Gamma}{dt} = \frac{\partial}{\partial x_i}\left(D \frac{\partial \Gamma}{\partial x_i}\right) + S \quad \rightarrow \quad \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D \frac{\partial k}{\partial x_j}\right) + S$

MODEL: $-\left(\frac{1}{\rho}\overline{u_j' p'} + \frac{1}{2}\overline{u_i' u_i' u_j'} - 2\nu \overline{u_i' s_{ij}}\right) = \left(\frac{\nu_T}{\sigma_k} + \nu\right) \frac{\partial k}{\partial x_j}$

Recalling: $-\overline{u_i' u_j'} = \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3} k \delta_{ij} \quad \rightarrow \quad -\overline{u_i' u_j'} S_{ij} = \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \frac{\partial \bar{u}_i}{\partial x_j}$

We named: $2\nu \overline{s_{ij} s_{ij}} = \varepsilon$

TURBULENCE MODELS

ONE EQUATION MODELS

TRANSPORT OF THE TURBULENT KINETIC ENERGY k:

$$\bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_k} + \nu\right)\frac{\partial k}{\partial x_j}\right] + \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon$$

We have not finished yet. We still need to know ν_T and ε.

σ_k is a coefficient to be tuned with experimental data.

TURBULENCE MODELS

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MODELING OF THE TURBULENT VISCOSITY :

We had: $\nu_T \propto \hat{V} L$

It seems natural to choose: $\hat{V} = \sqrt{k}$

Thus: $\boldsymbol{\nu_T = c'_\mu \sqrt{k} L}$ (Kolmogorov – Prandtl)

MODELING OF THE TURBULENT DISSIPATION RATE :

We said that the viscous dissipation depends on the dynamics of the large scales:

$$\varepsilon \sim \frac{\hat{V}^3}{L}$$

Thus: $$\boldsymbol{\varepsilon = c_D \frac{k^{3/2}}{L}}$$

TURBULENCE MODELS

ONE EQUATION MODELS

The equation for the transport of the turbulent kinetic energy is:

$$\bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_k} + \nu\right)\frac{\partial k}{\partial x_j}\right] + \nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - c_D \frac{k^{3/2}}{L}$$

With $\nu_T = c'_\mu \sqrt{k} L$

L is modelled with some algebraic model. For example, for shear layer flows:

$$L = y \quad , \quad \frac{y}{\delta} \le \frac{0.09}{\kappa}$$

$$L = 0.09\delta \quad , \quad \frac{y}{\delta} > \frac{0.09}{\kappa}$$

TURBULENCE MODELS

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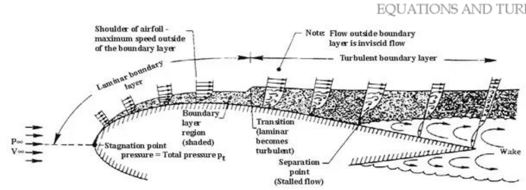
ONE EQUATION MODELS

BRADSHAW'S MODEL (1967, 1973)



https://www.quora.com/What-is-the-significance-of-a-boundary-layer

It was developed for **2D boundary layers** in which $-\overline{u'v'}/k = a_1 \approx \text{constant} \ (\approx 0.3)$.

The transport equation for k is transformed in a transport equation for $\overline{u'v'}$:

$$\bar{u}\frac{\partial}{\partial x}\left(\frac{\overline{u'v'}}{a_1}\right) + \bar{v}\frac{\partial}{\partial y}\left(\frac{\overline{u'v'}}{a_1}\right) = -\frac{\partial}{\partial y}\left[G\overline{u'v'}\sqrt{\overline{u'v'}_{MAX}}\right] - \overline{u'v'}\frac{\partial \bar{u}}{\partial y} - \frac{\overline{u'v'}^{3/2}}{L}$$

$$G = \frac{\sqrt{\overline{u'v'}_{MAX}}}{U_\infty} f_1\left(\frac{y}{\delta}\right) \quad , \quad \frac{L}{\delta} = f_2\left(\frac{y}{\delta}\right)$$

TURBULENCE MODELS

ONE EQUATION MODELS

Application of one equation models is restricted mainly to shear-layer flows because it is difficult to determine experimentally the length scale distribution in more complex flows.

TURBULENCE MODELS

TWO EQUATION MODELS

In the two equation models, another property is transported in addition to the turbulent kinetic energy, k. The most common models are:

- $k - \varepsilon$

- $k - \omega$, where $\omega = k/\nu_T$

(There are more, for example, shear stress transport (SST) model)

TURBULENCE MODELS

TWO EQUATION MODELS: $k - \varepsilon$

From dimensional analysis:

$$\nu_T \sim \frac{k^2}{\varepsilon} \qquad \rightarrow \qquad \nu_T = c_\mu \frac{k^2}{\varepsilon}$$

A transport equation for the energy dissipation rate ε is needed.

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TURBULENCE MODELS

TWO EQUATION MODELS: $k - \varepsilon$

TRANSPORT EQUATION FOR ε .

$$\varepsilon \equiv 2\nu\overline{s_{il}s_{il}} = \nu\overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_i'}{\partial x_l}}$$

It requires some boring algebra. The steps are the followings:

- Make a Reynolds decomposition of the Navier-Stokes Equation
- Get the equation for the instantaneous fluctuations
- Differentiate it with respect to x_l
- Multiply it by $2\nu\frac{\partial u_i'}{\partial x_l}$
- Take time average
- The exact transport equation for ε was obtained.

TURBULENCE MODELS

TWO EQUATION MODELS: $k-\varepsilon$

TRANSPORT EQUATION FOR ε .

The different terms arising in the equation for ε are modelled in analogous way as it was done for the turbulent kinetic energy. Finally, the transport equation for the energy dissipation rate is:

$$\bar{u}_j\frac{\partial \varepsilon}{\partial x_j}=\frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_\varepsilon}+\nu\right)\frac{\partial \varepsilon}{\partial x_j}\right]+c_{\varepsilon 1}\frac{\varepsilon}{k}\nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}-c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

TURBULENCE MODELS

TWO EQUATION MODELS: $k-\varepsilon$

THE SYSTEM OF DIFFERENTIAL EQUATIONS THAT SOLVE THE PROBLEM IS:

CONTINUITY EQUATION:

$$\frac{\partial \bar{u}_i}{\partial x_i}=0$$

REYNOLDS EQUATIONS:

$$\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{-\bar{p}}{\rho}\delta_{ij}+(\nu+\nu_T)\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\frac{2}{3}k\delta_{ij}\right)$$

TURBULENT VICOSITY:

$$\nu_T=c_\mu\frac{k^2}{\varepsilon}$$

TURBULENT KINETIC ENERGY TRANSPORT:

$$\bar{u}_j\frac{\partial k}{\partial x_j}=\frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_k}+\nu\right)\frac{\partial k}{\partial x_j}\right]+\nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}-\varepsilon$$

ENERGY DISSIPATION RATE TRANSPORT:

$$\bar{u}_j\frac{\partial \varepsilon}{\partial x_j}=\frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_\varepsilon}+\nu\right)\frac{\partial \varepsilon}{\partial x_j}\right]+c_{\varepsilon 1}\frac{\varepsilon}{k}\nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}-c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

TURBULENCE MODELS

TWO EQUATION MODELS: $k - \varepsilon$

VALUES OF THE PARAMETERS INVOLVED IN THE SYSTEM OF EQUATIONS

$$c_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\varepsilon = 1.3 \qquad c_{\varepsilon 1} = 1.44 \qquad c_{\varepsilon 2} = 1.92$$

These values are, unfortunately, not universal but have to be modified for other problems such as jets and wakes and recirculating flows (White)

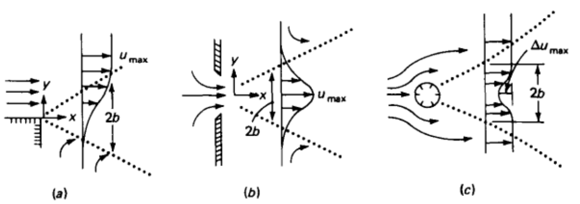
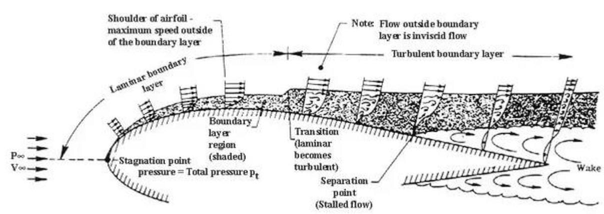


FIGURE 6-35
Three types of free turbulent flow: (*a*) mixing layer; (*b*) free jet; (*c*) wake of a body.

WHITE. Viscous Fluid Flows



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TURBULENCE MODELS

TWO EQUATION MODELS: $k - \varepsilon$. BOUNDARY CONDITIONS

Solid Boundaries

$$u^+ = \frac{1}{\kappa} ln(y^+) + B \qquad (30 < y^+ < 100) \qquad u^+ = \frac{\bar{u}}{u_*} \ , \qquad y^+ = \frac{y u_*}{\nu}$$

In the region $30 < y^+ < 100$, the Reynolds stresses are practically constants and equal to ρu_*^2.
Convection and diffusion $\overline{u'v'}$ are negligible $\Rightarrow$ $Production = Dissipation$:

Boundary condition for k*:*

$$\frac{k}{u_*^2} = \frac{1}{\sqrt{c_\mu}}$$

Boundary condition for ε*:*

$$\varepsilon = \frac{u_*^3}{\kappa y}$$

Free surface

Usually turbulent stresses, turbulent fluxes and ε are assumed equal to zero $\Rightarrow$ symmetry condition

TURBULENCE MODELS

TWO EQUATION MODELS: $\boldsymbol{k-\varepsilon}$. LOW REYNOLDS NUMBER EFFECTS

Modifications to the $\boldsymbol{k-\varepsilon}$ model presented before have been developed in order to take into account the effect of low Reynolds numbers. They include new parameters that have to be calibrated with the experimental data.

TURBULENCE MODELS

TWO EQUATION MODELS: $\boldsymbol{k-\omega}$

THE SYSTEM OF DIFFERENTIAL EQUATIONS THAT SOLVE THE PROBLEM IS:

In this model, the turbulent viscosity is given by: $\nu_T = \dfrac{k}{\omega}$

The transport equations for k and ω are:

$$\bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_k} + \nu\right)\frac{\partial k}{\partial x_j}\right] + \nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \beta^* k\omega$$

$$\bar{u}_j \frac{\partial \omega}{\partial x_j} = \frac{\partial k}{\partial x_j}\left[\left(\frac{\nu_T}{\sigma_\varepsilon} + \nu\right)\frac{\partial \omega}{\partial x_j}\right] + \alpha\frac{\omega}{k}\nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \beta\omega^2$$

$$\alpha = \frac{13}{25}\ , \quad \beta = \beta_0 f_\beta\ , \quad \beta^* = \beta_0^* f_\beta^*\ , \quad \sigma_k = 2\ , \quad \sigma_\omega = 2\ , \quad \beta_0 = \frac{9}{125}\ , \quad \beta_0^* = \frac{9}{100}$$

f_β is a function of the mean vorticity and mean shear deformation rate
f_β^* is a function of ω and the spatial derivatives of k and ω

TURBULENCE MODELS

TWO EQUATION MODELS

CONCLUSION

- $k-\varepsilon$: Valid for fully turbulent flows.

  Performs poorly for complex flows involving severe pressure gradient, separation, strong streamline curvature.

- $k-\omega$: Works well for wall-bounded and low Reynolds number flows.

  Suitable for complex boundary layer flows under adverse pressure gradient and separation. Can be used for transitional flows.

Turbulence models do not provide any new knowledge on turbulence. However, they are powerful tools for engineers.

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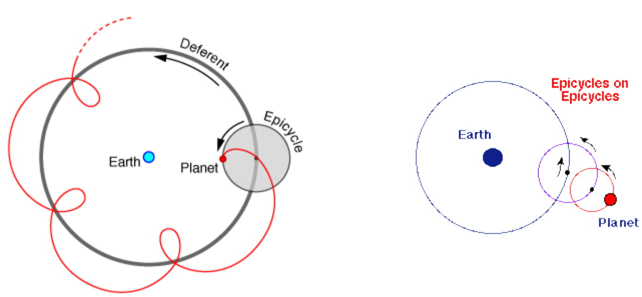
EQUATIONS AND TURBULENCE MODELS



THANKS!!

Questions?

TURBULENCE MODELS

THE CONCEPT OF TURBULENT VISCOSITY

REYNOLDS EQUATIONS:

$$\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{-\bar{p}}{\rho}\delta_{ij} + (\nu + \nu_T)\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij}\right)$$

TURBULENCE MODELS

ENSEMBLE AVERAGES??????

TURBULENCE MODELS

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TWO EQUATION MODELS: $\boldsymbol{k-\varepsilon}$

VALUES OF THE PARAMETERS INVOLVED IN THE SYSTEM OF EQUATIONS

$$c_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\varepsilon = 1.3 \qquad c_{\varepsilon 1} = 1.44 \qquad c_{\varepsilon 2} = 1.92$$

These values are, unfortunately, not universal but have to be modified for other problems such as jets and wakes and recirculating flows (White)

Axisymmetric jets: $c_\mu = 0.09 - 0.04f \quad , \quad c_{\varepsilon 2} = 1.92 - 0.0667f$

$$f = \left| \frac{\delta}{\Delta U_m} \left(\frac{\partial U_{max}}{\partial x} - \left| \frac{\partial U_{max}}{\partial x} \right| \right) \right|^{0.2}$$